Strong Inner Model Representations for Strong Cardinals: Set-Theoretic Universes Constructed Relative to Set-Sized Objects

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November 6, 2016

Abstract

This paper provides a self-contained exposition of the theory of inner models of the form $L[A]$ up to Kunen's result on the canonicity of inner models for measurable cardinals. This paper also shows how strong cardinals are incompatible with models of the form $L[A]$ for any set A. Necessary background material, including measurable and strong cardinals, direct limits, elementary embeddings, the model $L[A]$ and Scott style impossibility results are covered.

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Chapter 1

Introduction

Definitions, Basic Results

Relations

An *n-ary relation* R on X is a set of n-tuples such that $R \subseteq X^n$. A *binary relation* R is a 2-ary relation. For binary relations R, the *domain of* R is the set $\text{dom}(R) = \{w : \exists v(w, v) \in R\}$, and the *range of* R is the set $\text{dom}(R) = \{v : \exists w(w, v) \in R\}$.

Functions

A binary relation f is a *function* when $(x, y) \in f$ and $(x, z) \in f$ implies that $y = z$. For any function $f : x \to y$, so that f is of the form $\{\langle x_1, y_1\rangle, \ldots, \langle x_\alpha, y_\alpha\rangle, \ldots\}$ for $x_\alpha \in x, y_\alpha \in y$, for any $z \in x$, we define *the image of the object z under f*, denoted as $f(z)$, or simply fz, as the unique w such that $\langle z, w\rangle \in f$, if such as w exists, and undefined otherwise. For any $z \subset x$, we define *the image of the set z under f*, denoted as $f[z]$, or simply $f''z$, as the set $\{f(u)|u \in z\}$, and undefined when for no $u \in z$ do we have $j(u)$ defined.

Ordinals

An ordering $<$ is a *linear order on S*, or that $<$ *linearly orders* S, when for all $r, p, s \in S$, $s \not< s$ and $r < p < s$ implies that $r < s$. An element m of S is a $<$-least element of S when for all $s \in S$, $s \leq m$. We say that an order $<$ is a *well-ordering* of a set S, or that $<$ *well-orders* S, when $<$ linearly orders S and every non-empty subset $R \subseteq S$ has a $<$-least element.

A set S is called *transitive* when $y \in x \in S$ implies that $y \in S$. We can write this succinctly as $\bigcup S \subseteq S$. This definition is equivalent to saying that $x \in S$ implies that $x \subseteq S$, which we can write as $X \subseteq P(X)$.

An *ordinal* is a transitive set S that is well-ordered under $\in$, so that $\in$ in particular $\in$ is a linear ordering of S. We will denote ordinals by Greek lower case letter $\alpha, \beta, \gamma, \ldots$, and denote that α is an ordinal with $\alpha \in \mathrm{Ord}$. We will compare two ordinals α, β by saying that α is less than β, or $\alpha < \beta$, exactly when $\alpha \in \beta$.

Chapter 2

Direct Limits

2.1 Directed Diagrams and Direct Limits

This section will introduce directed diagrams and direct limits. Direct limits are an extremely useful technique for building models from systems of other models. Among other things, taking direct limits is an important technique in the theory of inner models of measurable cardinals. In particular, iterating the ultraproduct operation of models of set theory into the transfinite will require a limit stage definition. Direct limits provide exactly this.

Here we will describe how the properties of the direct limit depend on the properties of the original diagram, which is a partially ordered set of structures together with homomorphisms between the structures. We prove that when the maps in the diagram are embeddings, then so are the maps into the limit, which will be Theorem 4, and when they are elementary embeddings, then so are the maps into the limit, which will be Theorem 9. This presentation borrow from that of Hodges [3].

2.1.1 Definitions

We first recall the model-theoretic definitions we will need in the presentation to follow. Assume A and B are L-structures, and $f : \mathrm{dom}(A) \to \mathrm{dom}(B)$. Then f is a *homomorphism* when, for each constant symbol c, we have that:

$$c^B = f(c^A), \tag{2.1}$$

for each n-ary relation symbol R and n-tuple $\bar{a}$ from A, we have:

$$R^A(\bar{a}) \implies R^B(f\bar{a}), \tag{2.2}$$

and for each n-ary function symbol F and n-tuple $\bar{a}$ from A, we have:

$$f(F^A(\bar{a})) = F^B(f\bar{a}). \tag{2.3}$$

A map $f : \mathrm{dom}(A) \to \mathrm{dom}(B)$ is an *embedding* if and only f is an injective homomorphism with the additional bidirectional condition that, for any $\bar{a} \in \mathrm{dom}(A)$,

$$R^A(\bar{a}) \iff R^B(f\bar{a}). \tag{2.4}$$

An embedding $f : \mathrm{dom}(A) \to \mathrm{dom}(B)$ is an *elementary embedding* when, for each L-formula $\phi(\bar{x})$ and $\bar{a}$ from A,

$$A \models \phi(\bar{a}) \iff B \models \phi(f\bar{a}). \tag{2.5}$$

We will need the following Lemma that embedding preserve atomic formulas.

Lemma 1. *For L-structures A and B and a map $f : \mathrm{dom}(A) \to \mathrm{dom}(B)$, we have that f is an embedding if and only if for every atomic formula $\phi(\bar{x})$, and tuple $\bar{a}$ from $\mathrm{dom}(A)$, we have: $A \models \phi(\bar{a}) \iff B \models \phi(f\bar{a})$.*

Proof. (Idea) An induction from the definition of an embedding, together with the fact that, for homomorphisms f, any $\bar{a} \in A$ and term $t(\bar{x})$ of L, we have that:

$$f(t^A[\bar{a}]) = t^B[f\bar{a}]. \tag{2.6}$$

Full details can be found on p. 13 of Hodges[3]. □

Notation. Let L be a first-order language. Roman lower case letters $a, b, c, \ldots$ will refer to elements of models. For any L-structure A, we will write $a \in A$ instead of the official $a \in \mathrm{dom}(A)$ when there is no risk of ambiguity. When $\bar{a} = a_0, \ldots, a_n$, $f(\bar{a})$ will mean $f(a_0), \ldots, f(a_n)$, and the equivalence class $[\bar{a}]$ will mean $[a_0], \ldots, [a_n]$.

2.1.2 Directed Diagrams

We next develop the notion of a directed diagram. An *upward directed poset* is a partially ordered set $(P, \leq)$ such that if $m, n \in P$, then there exists a $p \in P$ with $m \leq p$ and $n \leq p$.

Definition 2 (*Directed Diagram*)**.** A *directed diagram in L* is an upward directed poset $(P, \leq)$, an L-structure A_m for each $m \in P$, and, for each pair $m, n \in P$ with $m \leq n$, a homomorphism $h_{mn} : A_m \to A_n$ such that:

$$h_{mm} \text{ is the identity map,} \tag{2.7}$$

$$m \leq n \leq p \text{ imply that } h_{mp} = h_{np} h_{mn}. \tag{2.8}$$

When we have that each h_{mn} is an elementary embedding, we will call the resulting diagram an *elementary directed diagram*. Assume for simplicity that in a directed diagram D, models A_m, A_n are disjoint when $m \neq n$.

We now turn towards defining the *direct limit* B, a new model constructed on top of the directed system D. Intuitively, the elements of B will the paths through the domains of the A_m's traced out by the homomorphisms given in the diagram. As we will show, every element of any A_m in D will belong to exactly one path. We will then define limit maps that take elements from each A_m to the unique path the element belongs to. The interpretations in B of the non-logical symbols of L will be given so that all these limit maps will be homomorphisms.

Definition 3 (*Direct Limit*)**.** The *direct limit* B of the directed diagram D will be the L-model $(\text{dom}(B), c^B, R^B, f^B)$, all of which we now define. For $\text{dom}(B)$, we first let $X := \bigcup_{p \in P} \text{dom}(A_m)$, the union of elements appearing in models A_m of the directed diagram. For $a, b \in X$, we define

$$\begin{aligned} a \sim b \quad \iff \quad & \text{there exist } m, n, t \in P \text{ such that } a \in \text{dom}(A_m), b \in \text{dom}(A_n), \\ & m \leq t, n \leq t \text{ and } h_{mt}(a) = h_{nt}(b). \end{aligned} \tag{2.9}$$

Hence $a \sim b$ when the threads that a and b are on eventually converge. Note that $\sim$ is an equivalence relation: reflexivity and symmetry are straightforward, while transitivity follows from the commutivity equation (2.8) above. We set the domain of the direct limit B as the set of equivalence classes of X under $\sim$:

$$\text{dom}(B) := \{[\alpha] : \alpha \in X\}, \tag{2.10}$$

where $[\alpha]$ is the equivalence class of α under $\sim$.

Notation. In what follows, when more precision is needed, Greek lower case letters $\alpha, \beta, \gamma, \dots$ will refer to elements of the models A_m, while Roman letters $a, b, c, \dots$ will be reserved for elements of the direct limit B, which is to be defined below.

For the interpretations in B of the non-logical symbols of L, we first define, for each $m \in P$, the *limit map* $h_m : \mathrm{dom}(A_m) \to \mathrm{dom}(B)$ by setting:

$$h_m(\alpha) = [\alpha]. \tag{2.11}$$

Then for c a constant symbol of L, we pick any $m \in P$ and let:

$$c^B := h_m(c^{A_m}), \tag{2.12}$$

for R a relation symbol of L and a tuple $\bar{b}$ of $\mathrm{dom}(B)$, we set:

$$\bar{b} \in R^B \iff \exists p \in P, \exists \bar{a} \in \mathrm{dom}(A_m) \text{ such that } h_m(\bar{a}) = \bar{b} \text{ and } R^{A_m}(\bar{a}), \tag{2.13}$$

and for f a function symbol of L and a tuple $\bar{b}$ of $\mathrm{dom}(B)$, we have at least one $m \in P$ such that $h_m(\bar{a}) = \bar{b}$ for some $\bar{a} \in \mathrm{dom}(A_m)$ by the upward directedness of P, hence we let:

$$f^B(\bar{b}) := h_m(f^{A_m}(\bar{a})). \tag{2.14}$$

Notice that as each h_{mn} is a homomorphism and the homomorphisms of a directed system commute, these definitions will be sound, and that these definitions for truth in B are chosen so that each limit map h_m is a homomorphism into B, which we show in the next section. In the next section we will also show that special features of the h_{mn}'s in D will propagate to the limit maps. In particular, if all h_{mn}'s are embeddings, then each limit map will be an embedding, and if all h_{mn}'s are elementary embeddings, then each limit map will be an elementary embedding.

2.2 Main Theorem on Direct Limits

We now set out to prove the Main Theorem of this chapter on directed systems and direct limits.

Theorem 4 (Main Theorem). 1. *The definitions of a directed system are all sound.*

2. *The limit maps $h_m : A_n \to B$ of directed systems commute with the given maps h_{mn}, so that $h_m = h_n h_{mn}$, whenever $m \leq n$.*

3. *Whenever the maps h_{mn} are all embeddings, then the maps h_m are also all embeddings.*

4. *Whenever the maps h_{mn} are all elementary embeddings, then the maps h_m are also all elementary embeddings.*

We will proceed by proving these respectively as the Soundness, Limit Map, Embedding, and the Elementary Embedding Lemmas.

2.2.1 Soundness Lemma, Limit Map Lemma

Here we prove the soundness of the definitions for the limit structure, as well as the fact that the limit maps are indeed homomorphisms.

Lemma 5 (**Soundness Lemma**). *For any L-function symbol F or L-constant symbol c, the definitions of $F^B(\bar{b})$ and c^B are sound, so that $F^B(\bar{b})$ does not depend on the choice of p, $\bar{a}$ above, and c^B does not depend on the choice of p.*

To prove this, we first we say that tuples $\bar{a} \sim \bar{b}$ exactly when $\bar{a}$ and $\bar{b}$ are the same length and $a_i \sim b_i$ for every i less than the length of $\bar{a}$. Then we can find a uniform top model where the threads of each a_i and b_i nicely converge.

Lemma 6. *For any $\bar{a} \in A_m, \bar{b} \in A_n$, we have:*

$$\bar{a} \sim \bar{b} \iff \text{there exists } t \in P \text{ such that } m, n \leq t \text{ and } h_{mt}(\bar{a}) = h_{nt}(\bar{b}). \tag{2.15}$$

Proof. Right to left is by definition of $\bar{a} \sim \bar{b}$, so for the other direction, we have models $A_{p_0}, \ldots A_{p_{n-1}}$ that witness $a_i \sim b_i$ for $0 \leq i < n$. By repeated use of upward directedness of P on these models, we find a uniform t where the images of $\bar{a}$ and $\bar{b}$ converge. □

Next we show the Limit Map Lemma towards the Soundness Lemma.

Lemma 7 (**Limit Map Lemma**). *For any $m, n \in P$ such that $m \leq n$, we have:*

$$h_m = h_n h_{mn}. \tag{2.16}$$

Proof. Let m, n as above, $\alpha \in A_m$. We have $h_{mn}(\alpha) \sim \alpha$ by the definition of similarity, which means $[h_{mn}(\alpha)] = [\alpha]$, whence $h_m(\alpha) = h_n h_{mn}(\alpha)$ for any $\alpha \in A_m$, as desired. □

Now we can prove the soundness of the direct limit. Experienced readers may feel free to skip this proof without losing the main thread of the narrative.

Proof of the Soundness Claim. Let $\bar{b} \in B$, and assume $m, n \in P$ with $\bar{\alpha} \in A_m$, $\bar{\beta} \in A_n$ such that:

$$h_m(\bar{\alpha}) = h_n(\bar{\beta}) = \bar{b}. \tag{2.17}$$

Both $\bar{\alpha}$ and $\bar{\beta}$ are in the same equivalence class, hence by the first Lemma, we have some $t \geq m, n \in P$ such that:

$$h_{mt}(\bar{\alpha}) = h_{nt}(\bar{\beta}) \tag{2.18}$$

From this we derive that F^B is well-defined:

$$\begin{aligned} h_m(F^{A_m}(\bar{\alpha})) &= h_t h_{mt} F^{A_m}(\bar{\alpha}) \\ &= h_t F^{A_t}(h_{mt}(\bar{\alpha})) \\ &= h_t F^{A_t}(h_{nt}(\bar{\beta})) \\ &= h_t h_{nt} F^{A_n}(\bar{\beta}) \\ &= h_n(F^{A_n}(\bar{\beta})). \end{aligned}$$

The first and last equalities follow by the second Lemma above, the second and fourth from h_{mt}, h_{nt} homomorphisms, while the third from equation (2.18). This shows that F^B is well-defined.

We claim that c^B is well-defined also, so that, for any $m, n \in P$, we have:

$$h_m(c^{A_m}) = h_n(c^{A_n}) \tag{2.19}$$

By the upward directedness of P, there exists $t \in P$ such that $m, n \leq t$ with homomorphism h_{mt} (resp. h_{nt}) from A_m (resp. A_n) into A_t. From this, we conclude:

$$h_m(c^{A_m}) = c^{A_t} = h_n(c^{A_n}) \tag{2.20}$$

Hence the definition of c^B is well-defined for any L-constant symbol. □

2.2.2 Embedding Lemma

Lemma 8 (**Embedding Lemma**). *In any L-directed diagram, when each map is an embedding, then each limit homomorphism will also be an embedding.*

Proof. By above, $h_m : A_m \to B$ is an embedding if and only h_m is an injective homomorphism with the strengthened condition that for any $\bar{a} \in \text{dom}(A_m)$ we have:

$$R^{A_m}(\bar{a}) \iff R^B(h_m(\bar{a})) \tag{2.21}$$

For injectivity, let $m \in P$. If h_m were not injective, then there would exist $a, b \in \text{dom}(A_m)$ such that $a \neq b$ and $h_m(a) = h_m(b)$, which implies that $[a] = [b]$, and hence $a \sim b$. But $a \sim b$ is equivalent to the existence of a $t \in P$ such that $m \leq t$ and $h_{mr}(a) = h_{mr}(b)$, by the definition of $\sim$. As by assumption h_{mt} is an embedding (so, in particular, injective) we have that $h_{mt}(a) = h_{mt}(b) \iff a = b$. This is a contradiction with our assumption that $a \neq b$. Hence h_m is injective for any $m \in P$.

For (2.21), we claim:

$$\begin{aligned} R^{A_m}(\bar{\alpha}) &\iff \exists r \in P, \bar{\beta} \in A_r \text{ such that } R^{A_r}(\bar{\beta}) \text{ and } h_r(\bar{\beta}) = [\bar{\alpha}] \\ &\iff R^B([\bar{\alpha}]) \\ &\iff R^B(h_m(\bar{\alpha})) \end{aligned}$$

For the rightward direction, the first equivalence follows by taking m and $\bar{\alpha}$ as the witnessing $r \in P$, $\bar{\beta} \in A_r$, while the second and third equivalences are by the definitions of R^B, h_m respectively.

For the leftward direction, assume $R^B(h_m(\bar{\alpha}))$, so that we have some $r \in P$, $\bar{\beta} \in A_r$ such that $R^{A_r}(\bar{\beta})$ and $h_r(\bar{\beta}) = [\bar{\alpha}]$. Equivalently, $\bar{\alpha} \sim \bar{\beta}$. From Lemma 6, we have a $t \geq r, m \in P$ such that $h_{mt}(\bar{\alpha}) = h_{rt}(\bar{\beta})$, so that:

$$\begin{aligned} R^{A_r}(\bar{\beta}) &\iff R^{A_t}(h_{rt}(\bar{\beta})) \\ &\iff R^{A_t}(h_{mt}(\bar{\alpha})) \\ &\iff R^{A_m}(\bar{\alpha}), \end{aligned}$$

Here the first and third equivalences follow from the facts that h_{rt}, h_{mt} are embeddings, where $h_{mt}(\bar{\alpha}) = h_{rt}(\bar{\beta})$ is by choice of t.

Hence we conclude $R^{A_m}(\bar{\alpha})$, completing the leftward direction. Therefore we have (2.21), and hence each h_m is an embedding, as desired. □

2.2.3 Elementary Embedding Lemma

Lemma 9 (**Elementary Embedding Lemma**). *In any L-directed diagram, when each map is an elementary embedding, then each limit homomorphism will also be an elementary embedding.*

Proof. Denote by j_m the limit homomorphism from A_m to B. We prove, by induction on complexity of formula ϕ, that each j_m is elementary, so that, for every $m \in P$ and every $\bar{\alpha} \in A_m$ we have the elementary embedding schema:

$$A_m \models \phi(\bar{\alpha}) \iff B \models \phi(j_m(\bar{\alpha})). \tag{2.22}$$

Assume ϕ atomic, $m \in P$ and $\bar{\alpha} \in A_m$. By Theorem 1, we have that (2.22) holds for ϕ atomic if and only if j_m is an embedding. But the fact that j_m is an embedding for any $m \in P$ is shown above in (iii). Hence (2.22) holds for atomic ϕ, while Boolean cases are verified as always.

Finally assume $\phi(\bar{\alpha})$ is $\exists x \psi(\bar{\alpha}, x)$, with (2.22) holding for ψ, and $\bar{\alpha}$, m as above. For the leftward direction, assume $B \models \phi(j_m(\bar{\alpha}))$, so that there exists some $c \in B$ such that $B \models \psi(j_m(\bar{\alpha}), c)$. Picking representatives of each coordinate of the vector $j_m(\bar{\alpha}), c$ and applying the upward directedness of P, we have an $r \in P$, $\bar{\gamma}_1, \gamma_2 \in A_r$ such that:

$$j_r(\bar{\gamma}_1, \gamma_2) = j_m(\bar{\alpha}), c. \tag{2.23}$$

Therefore $B \models \psi(j_r(\bar{\gamma}_1, \gamma_2))$. Applying the Induction Hypothesis to the model A_r and vector $\bar{\gamma}_1, \gamma_2$ gives:

$$B \models \psi(j_r(\bar{\gamma}_1, \gamma_2)) \iff A_r \models \psi(\bar{\gamma}_1, \gamma_2). \tag{2.24}$$

Hence $A_r \models \psi(\bar{\gamma}_1, \gamma_2)$, so that $A_r \models \phi(\bar{\gamma}_1)$. As $\bar{\gamma}_1 \sim \bar{\alpha}$, by Lemma 6, we have a $t \in P$ such that $h_{rt}(\bar{\gamma}_1) = h_{mt}(\bar{\alpha})$. Since the maps h_{mt}, h_{rt} are, by assumption, elementary embeddings, we have:

$$\begin{aligned} A_r \models \phi(\bar{\gamma}_1) &\iff A_t \models \phi(h_{rt}(\bar{\gamma}_1)) \\ &\iff A_t \models \phi(h_{mt}(\bar{\alpha})) \\ &\iff A_m \models \phi((\bar{\alpha})). \end{aligned}$$

Then $A_m \models \phi((\bar{\alpha}))$, completing the leftward direction.

Rightwards, assume β is the witness such that $A_m \models \exists x \psi(\bar{\alpha}, x)$. It is straightforward to verify that $j_m(\beta)$ is the witness in B to the formula $\exists x \psi(j_m(\bar{\alpha}), x)$, so that $B \models \phi(j_m(\bar{a}))$, as desired. This completes the proof of the Elementary Embedding Lemma, and hence our Main Theorem. $\square$

2.2.4 The Tarski-Vaught Theorem

Taking the union of a chain of structures is a ubiquitous construction in model theory, with applications including amalgamation constructions (see Chapter 6 of [3]) and building saturated models (Chapter 10 of [3]). Here we consider unions of chains as special kinds of directed diagrams, in particular, directed diagrams whose underlying order is an ordinal. We conclude by proving a theorem by Alfred Tarski and Robert Vaught on unions of elementary chains Theorem 13 as a corollary of our Main Theorem of this section.[1]

Definition 10. For any L and ordinal γ, let $(A_i : i < \gamma)$ a sequence of L-structures. Then $(A_i : i < \gamma)$ is called a *chain* when for all $i < j < \gamma$, we have $A_i \subseteq A_j$, that is, A_i is a substructure of A_j. For any chain, we define the *union of the chain* $\bigcup_{i<\gamma} A_i$ as the structure B with:

1. $\text{dom}(B) = \bigcup_{i\in\gamma} \text{dom}(A_i)$,
2. for $\bar{b} \in \text{dom}(B)$, let $R^B(\bar{b}) \iff (\exists i < \gamma) R^{A_i}(\bar{b})$,
3. for $\bar{b} \in \text{dom}(B)$, $\bar{b}$ appears in some A_i for $i < \gamma$, hence let $F^B(\bar{b}) = F^{A_i}(\bar{b})$,
4. $c^B = c^{A_i}$ for any $i < \gamma$.

As with the direct limit of a directed diagram, it is not hard to show the soundness of these definitions. Note that the union of a chain $(A_i : i < \gamma)$ is isomorphic to the direct limit of the directed diagram whose underlying order is γ, structures A_i at each $i < \gamma$, and inclusion maps between A_i and A_j when $i < j$.

Definition 11. An *elementary* chain is one where each inclusion is elementary, that is, the inclusion map i preserves all first-order formulas.

We are now in a position to prove the Tarski-Vaught Theorem on the union of elementary chains. First we state and show:

Lemma 12. *Disregarding differences up to isomorphism, any L-directed diagram, when each map is an inclusion, then each limit homomorphism will also be an inclusion.*

[1] This subsection may be skipped without interrupting the main flow of the work.

Proof. If each map is such that, for $i < j$, we have $A_i \subseteq A_j$, then each thread will contain exactly one element. Then the limit map will take any element from any A_i to the equivalence class containing just that one element, which we simply equate with that element itself. □

Tarski-Vaught theorem on unions of elementary chains 13. *Let $(A_i : i < \gamma)$ be an elementary chain of L-structures. Then the union $\bigcup_{i<\gamma} A_i$ of the chain is an elementary extension of A_i for each $i < \gamma$.*

Proof. Let $(A_i : i < \gamma)$ be as above. We have that γ is a linear order, hence a partial order, so that $(A_i : i < \gamma)$ together with the inclusion maps between each pair of models is an elementary directed diagram. We then apply our Lemma 12 above to conclude that each limit map j_m is an inclusion and our Main Theorem 9 to conclude that each j_m is elementary. Hence the union of the chain is an elementary extension of A_i, for each $i < \gamma$, as desired. □

In this chapter, we introduced directed systems and direct limits, proved our Main Theorem on direct limits, and showed the Tarski-Vaught Theorem on elementary chains as a corollary of our Main Theorem. In the next chapter we turn to measurable, inner models of which will are built using direct limits.

Chapter 3

Measurable Cardinals

This chapter introduces measurable cardinals and some of their most important properties. We exposit a number of results that illustrate the extremely interesting and subtle connections between these cardinals and non-trivial mappings between the universe of sets V and models of set theory. In particular, this chapter will present the *Fundamental Theorem of Ultrapower Models*: There is a measurable cardinal if and only if there is a non-trivial elementary embedding of V into an inner model of ZFC.

3.1 Measurable Cardinals

Measurable cardinals were introduced by the Polish mathematician Stanisław Ulam in 1930, who proved that measurable cardinals must be strongly inaccessible. In the late 1950s and early 1960s, Jerome Keisler and Dana Scott gave a proof of the Fundamental Theorem of Ultrapower Models. This Fundamental Theorem set a new paradigm for subsequent work in large cardinal theory that continues to current research: There is a large cardinal κ if and only if there exists a non-trivial elementary embedding between inner models with set of properties P. Measurables provide the base case when P is empty. Hence measurable cardinals are an important, perhaps even the central, large cardinal hypothesis. This characterization of measurable cardinals in terms of mappings has in turn has lead to a number of fruitful generalizations such as strong and supercompact cardinals, ones which we introduce in later chapters. The results by Ulam, Keisler and Scott on mesaruables are exposited in this Chapter below.

3.1.1 Definitions

We provide some basic definitions and notions that will be important for our later work on measurable cardinals.

σ-Algebras

For X a set, a collection A of subsets of X is called a *σ-algebra in X* if A satisfies:

1. $X \in A$
2. If $Y \in A$, then $Y^c \in A$.
3. If $Y_i \in A$ for $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} \in A$.

Notice that the first two properties imply that $\emptyset \in A$, which together with the last property implies that A is closed under countable unions. Since the complement of a union of complements is an intersection, then A is also closed under countable intersections.

Measures

Let A be a σ-algebra of subsets of a set X. Then a measure μ on A is a function $\mu : A \to [0, \infty]$ satisfying the following property:

- $\mu(\bigcup_i X_i) = \Sigma_i \mu(X_i)$ for any countable collection $\{X_i\}_{i \in \mathbb{N}}$ of disjoint sets in A.

We assume that there is at least one set $S \in A$ such that $\mu(S) < \infty$ to avoid trivialities. Notice that if we have one set S such that $\mu(S) < \infty$, then $\mu(\emptyset) = 0$, and our countable additivity condition implies that if $Y, Z \in A$ and $Y \subset Z$, then $\mu(Y) \leq \mu(Z)$, and in particular that for all $Y \in A$, $\mu(Y) \geq 0$.

If in addition $\mu(X) = 1$, then we call our measure *probabilistic*. If for any $x \in X$ such that $\{x\} \in A$, we have that $\mu(\{x\}) = 0$, we call our measure *non-trivial*.

As a particular example of a σ-algebra, we will often consider the full power set $P(S)$ of a set S, and when no danger of confusion exists, we will call μ a *measure on* S when it officially it is a measure on $P(S)$.

Filters

A *filter* F on a set X is a collection of subsets of X so that for each $Y, Z \subseteq X$, we have:

1. The empty set is not in F, and X is in F.

2. If Y is in F, $Y \subseteq Z$, then Z is in F also.

3. F is closed under finite intersections.

A filter is said to be an *ultrafilter* when in addition, we have that for any $Y \subset X$ either Y or $X - Y$ is in F. We call a filter F on X *principal* when $F = \{Y | Y \subseteq X\}$ for some subset $Y \subseteq X$ and *non-principal* otherwise. For β an ordinal $\leq |X|$, we further call a filter F to be *β-complete* when any collection of less than β many elements of F implies that the intersection of this elements is also in F. Note that the last clause of the definition of a filter, every filter is ω-complete. In fact it is not too hard to show that non-trivial 0-1 measures are equivalent to non-principal ultrafilters. We prove the following simple proposition that will be useful later on.

The Measure Problem

We can ask the question, are there any non-trivial extensions of the Lebesgue measure from the Lebesgue measurable sets to the full power set of $\mathbb{R}$? Or are there any non-trivial measures at all on the power set of $\mathbb{R}$? Are there any non-trivial measures on any uncountable set S at all? It turns out that if there is any measure on an uncountable set at all, then either there is an atomless set of cardinality less than or equal to the continuum, or there is a measure with an atom, and that measure is on a measurable cardinal.

Measurable Cardinals

Now we can define a measurable cardinal:

Definition 14. An uncountable cardinal κ is *measurable* when it has a non-principal κ-complete filter.

To show how a measure can have an impact on the structure of a set, we show the following proposition in full detail.

Proposition 15. *For κ measurable, X a subset of κ, $\mu(x) = 1$ implies that X has cardinality κ.*

Proof. Let $X \subset k$, $\mu(X) = 1$. Further assume for contradiction that $|X| < \kappa$. Consider any $y \in X$. As $X \subset k$, $y \in k$. As μ is non-trivial, $\mu(\{y\}) = 0$. But we also have that X is the disjoint union of its elements:

$$X = \bigcup_{i<\kappa} y_i \text{ such that } y_i \in X. \tag{3.1}$$

Hence, by κ-additivity of μ, we have:

$$\mu(X) = \sum_{i<\kappa} \mu(y_i) = \sum_{i<\kappa} 0 = 0. \tag{3.2}$$

This contradicts our assumption that $\mu(X) = 1$. Hence $|X| = \kappa$. □

Note that the converse that sets of size κ must be in the measure does not hold, since for any set X such that both X and $\kappa - X$ are of size κ, one of either X and $\kappa - X$ cannot be in the filter, by additivity.

Normality

Definition 16. For a filter F over λ, F is *normal* if for any $\langle X_\alpha | \alpha < \kappa \rangle \in {}^\lambda F$, its diagonal intersection $\Delta_{\alpha<\kappa} = \{\, \zeta < \lambda \mid \zeta \in \bigcap_{\alpha<\zeta X_\alpha} \,\} \in F$.

In fact an ultrafilter U is normal over λ if and only if for every function f regressive on a set $X \in U$, we have that f is constant on a set $Y \in U$.

Proposition 17. *For a κ-complete ultrafilter U over an uncountable κ, the following are equivalent:*

1. *U is normal.*
2. *$[d]_U = \kappa$, where $d : \kappa \to \kappa$ is the identity map on κ.*
3. *For every $X \subseteq \kappa$, $X \in D$ if and only if $\kappa \in j_D(X)$.*

Proof. (i) implies (ii): that $\kappa \subseteq [d]$ is basic. Hence let $[f] \in [d]$, so that f is a.e. regressive. Then $f(\alpha) = \zeta$ for a.e. α, so that $[f] = [c_\zeta] = j(\zeta) = \zeta \in \kappa$, and hence $[d] \subseteq \kappa$.

(ii) implies (iii): For $X \subseteq \kappa$, $X \in D$ if and only if $d(\alpha) \in X$ a. e., or equivalently $\kappa = [d] \in j_D(X)$.

(iii) implies (i): Let f be a.e. regressive, so that $\kappa \in j(\{ x \in \kappa \mid f(x) \in x \})$, which is just $j(f)(\kappa) \in \kappa$. Say $j(f)(\kappa) = \gamma < \kappa$, so that f is a.e. constant. □

Proposition 18. *For any measurable κ, there exists a normal ultrafilter over κ.*

Proof. As in Exercise 5.12 of [1], let U witness that κ measurable, f such that $[f]_U = \kappa$, and $F = \{ X \subseteq \kappa \mid f^{-1}(X) \in U \}$. Assume $\langle X_\alpha | \alpha < \kappa \rangle \in {}^\kappa F$, so that for any $\alpha < \kappa$, $f(x) \in X_\alpha$ for a.e. $x \in \kappa$. Then $f(x) \in \bigcap_{\alpha<\kappa} X_\alpha$ for a.e. $x \in S$, by κ-completeness of U, so that $f^{-1}(\Delta_{\alpha<\kappa} X_\alpha) \in U$, and hence $\Delta_{\alpha<\kappa} X_\alpha \in F$. □

3.1.2 Results

The next theorem due to Ulam demonstrates that measurable cardinals are indeed large:

Theorem 19. *For any uncountable κ, κ measurable implies that κ is strongly inaccessible.*

Proof. Regularity: Assume for contradiction that there exists a limit ordinal λ such that the sequence $\langle \alpha_i | i < \lambda < \kappa \rangle$, is cofinal in κ. By Lemma 15, for each i, as $|\alpha_i| \neq \kappa$, we have $\mu(\alpha_i) \neq 1$, so that $\mu(\alpha_i) = 0$. We express κ as a union of disjoint sets, for any $i < \lambda$ by defining:

$$\bar{\alpha}_i := \begin{cases} \alpha_i \backslash \{\alpha_i - 1\} & \text{when } \alpha \text{ is a successor, and} \\ \varnothing & \text{otherwise} \end{cases}$$

It is straightforward then that: $\bigcup_{i<\lambda<\kappa} \bar{\alpha}_i = \kappa$, $\mu(\bar{\alpha}_i) = 0$ (as $\bar{\alpha}_i \subset \alpha$) for each i, and that κ-additivity implies that $\mu(\kappa) = 0$, a contradiction.

Strong Limit: Assume for contradiction that there exists a $\lambda < \kappa$ such that $\kappa \leq 2^\lambda$. Let $F \subset {}^\kappa 2$ such that $|F| = \kappa$. Let U witness that κ is measurable. Define for each $\alpha < \Lambda$:

$$X_i := \begin{cases} \{f \in F | f(\alpha) = 1\} & \text{if } \{f \in F | f(\alpha) = 1\} \in U, \text{ and} \\ \{f \in F | f(\alpha) = 0\} & \text{otherwise,} \end{cases}$$

and

$$\varepsilon_\alpha := 1 \iff X_\alpha = \{f \in F | f(\alpha) = 1\}. \tag{3.3}$$

That U is κ-complete implies that $\bigcap_{\alpha<\kappa} X_\alpha \in U$. But $\bigcap_{\alpha<\kappa} X_\alpha \in U$ is just the singleton containing the function $f(\alpha) = \varepsilon_\alpha$. Contradiction with U non-trivial. □

Chapter 4

Measurable Cardinals, Elementary Embeddings

4.1 Elementary Embeddings j

The modern language of large cardinals is largely given in terms of certain maps called elementary embeddings. Here we present the definitions surrounding elementary embeddings, inner models, and some simple results connecting the two.

Definition 20. For two L-structures A and B, a map $j : \mathrm{dom}(A) \to \mathrm{dom}(B)$ is an *elementary embedding* when, for each L-formula $\phi(\bar{x})$ and $\bar{a}$ from A, we have:

$$A \models \phi(\bar{a}) \iff B \models \phi(j(\bar{a})) \tag{4.1}$$

Theorem 21 (Gaifman). *There is a measurable cardinal if and only if there is a non-trivial elementary embedding j from V to a transitive model M of set theory.*

Definition 22. By an Σ_n-*embedding*, denoted as $j : A \prec_n B$, we will mean that j is an elementary embedding for Σ_n^{ZF} formulas. By taking negations, we have that Σ_n-embeddings are elementary for Π_n^{ZF} formulas as well.

Inner Models

Definition 23. The term *inner model* will mean a transitive model of ZFC containing all ordinals such that the relation symbol $\in$ is interpreted as the true element of relation. Inner models will be denoted as M, M_0, M_1, and so on.

To give a sense of how elementary embeddings generate a structure theory of inner models, we present the details of the following basic proposition.

Proposition 24. *Suppose that M_0 and M_1 are inner models and $j_0 : M_0 \prec_1 M_1$. Then for any ordinal α, we have that for any ordinal α, $j(\alpha)$ is an ordinal $\leq \alpha$.*

Proof. The formulas Trns(x) and LinOrd(x) that hold exactly when x is respectively transitive and linearly ordered are both Σ_0^{ZF}. Hence $j(\alpha)$ is an ordinal by the fact that j is elementary. We prove that $\alpha \leq j(\alpha)$ by induction on Ord:

For $\alpha = \emptyset$, then $M_0 \models \phi(\alpha)$, where

$$\phi(x) := \forall y(y \in x \rightarrow y \neq y). \tag{4.2}$$

Hence $M_1 \models \phi(j(\alpha))$ by j elementary, so that $j(\alpha)$ is $\emptyset$, whence $\alpha = j(\alpha)$, and hence $\alpha \leq j(\alpha)$.

For α a limit ordinal, assume for contradiction that $j(\alpha) < \alpha$. Then by the induction hypothesis, $\gamma \leq j(\gamma)$ for all $\gamma < \alpha$, so in particular,

$$j(\alpha) \leq j(j(\alpha)). \tag{4.3}$$

But taking $\psi(x, y) := x < y$, we have that $M_0 \models \psi(j(\alpha), \alpha)$. Then $M_1 \models \psi(j(j(\alpha)), j(\alpha))$ by j elementary, whence

$$j(j(\alpha)) < j(\alpha). \tag{4.4}$$

This is a contradiction with equation (4.3), so that $j(\alpha) \leq \alpha$, as desired.

For α successor, say that $\alpha = \beta + 1$, with the claim holding for all $\gamma < \alpha$. In particular, as $\beta < \alpha$, we have that

$$\beta \leq j(\beta) \tag{4.5}$$

With Succ$(x, y) := y = x \cup \{x\}$, we have that $M_0 \models$ Succ(β, α). By j elementary, we have that

$$M_1 \models \text{Succ}(j(\beta), j(\alpha)). \tag{4.6}$$

Combining, we get

$$\alpha = \beta \cup \{\beta\} \leq j(\beta) \cup \{j(\beta)\} = j(\alpha), \tag{4.7}$$

where the fist $=$ is by assumption, the second $=$ by equation (4.6), and the middle $\leq$ by equation (4.5). Hence $\alpha \leq j(\alpha)$, as desired. Note that all formulas we applied the assumption that $j : M_0 \prec_1 M_1$ were Σ_0^{ZF}. $\square$

Critical Points

We define the least ordinal κ such that $\kappa < j(\kappa)$ as the *critical point* of j of M, denoted as $\mathrm{crit}(j)$. It can in fact be shown that if j is elementary, then j will always have a critical point. Critical points play an important role in results below, and will always be large cardinals.

Note that for a critical point κ of an embedding, both $j(\kappa)$ and $j[\kappa]$ are defined. On the one hand, by definition, $j(\kappa) > \kappa$. On the other hand

$$j[\kappa] := \{j(u)|u \in \kappa\} = \{u|u \in \kappa\} = \kappa \tag{4.8}$$

so that $j(x) \neq j[x]$ in general.

4.2 Ultraproducts and Ultrapowers

Here we introduce the ultraproduct construction and fix the notation to be used, with an eye towards constructing an ultraproduct of V. In this section, we will follow the presentation as in p. 9 and pp. 47-9 of [1].

The Ultraproduct Construction

Let U be an ultrafilter on a set S. For each $i \in S$, let $\mathcal{M}_i = \langle M_i, \dots \rangle$ be an $\mathcal{L}$-structure. Let $\prod_S M_i$ denote the product of the M_i's, that is, the set of all functions f with domain S such that $f(i) \in \mathrm{dom}(M_i)$. For $f, g \in \prod_S M_i$, we let

$$f =_U g \iff \{i \in S | f(i) = g(i)\} \in U. \tag{4.9}$$

Measure theoretic terminology will be useful, so that $f =_U g$ will hold when $f(i) = g(i)$ on almost all i, with U providing the notion of largeness.

We can check that $=_U$ is an equivalence relation, so we can define $(f)_U$, or simply (f) when context is clear, as the equivalence class of f. We let $\prod_S M_i \setminus U := \{(f)|f \in \prod_S M_i\}$. We define the *ultraproduct of the $\mathcal{M}_i$'s by U* as the $\mathcal{L}$-structure with domain $\prod_S M_i$, which interprets the n-ary relation R as R_U defined by

$$\langle (f_1), \dots, (f_n) \rangle \in R_U \iff \{i \in S | \langle f_1(i), \dots, f_n(i) \rangle \in R_i\} \in U. \tag{4.10}$$

The interpretations of any function or constant symbols are defined analogously with respect to what happens ultrafilter many times.

Note that for language of set theory structure, if in each $\mathcal{M}_i$, E is the real membership relation, then

$$\langle (f), (g) \rangle \in E_U \iff \{i \in S | f(i) \in g(i)\} \in U. \tag{4.11}$$

Łoś' Theorem

We have the following important result on ultraproduct models:

Theorem 25 (Łoś)**.** *For any formula ϕ and $\vec{f}$ from $\prod_S M_i$, we have*

$$\prod_S \mathcal{M}_i \setminus U \models \phi((f_1), \dots, (f_n)) \iff \{i | \mathcal{M}_i \models \phi(f_1(i), \dots, f_n(i))\} \in U. \tag{4.12}$$

In other words, an arbitrary formula ϕ with parameters holds in the ultraproduct model exactly when ϕ holds in ultrafilter many of the base models $\mathcal{M}_i$'s on representatives of the parameters.

Proof. An induction on complexity of ϕ. The most important cases are negation, where we use that one of X or $S \setminus X$ is in U for any $X \subseteq S$, and the existential case, where we use the Axiom of Choice. □

4.2.1 Ultrapower Models of ZFC

When there is a fixed $\mathcal{M} = \langle M, \dots \rangle$ such that $\mathcal{M} = \mathcal{M}_i$ for each i, then the ultraproduct is called the *ultrapower of* $\mathcal{M}$ *by* U and denoted as ${}^S\mathcal{M} \setminus U$. For our purposes, the set S will be a cardinal κ, $\mathcal{L}$ will be the language of set theory, and $\mathcal{M}_i = V$ for every $i \in \kappa$.

Scott's Trick

We wish to construct an ultrapower of V, but face the problem that for any $f : S \to V$, we have that in general, $(f)_U$ is a proper class, so not formalizable in a first-order way. Using a trick from Dana Scott however, we can bypass this issue by defining

$$(f)_U^0 := \{g | g \in (f)_U \wedge \forall h (h \in (f)_U \to \text{rank}(g) \leq rank(h)))\}. \tag{4.13}$$

or in other words, functions in the equivalence class of f of minimal rank. This $(f)^0_U$ is a set, so that we can let the domain of the ultraproduct be the class

$$^{\kappa}V \setminus U := \{(f)^0_U | f : \kappa \to V\}, \tag{4.14}$$

and define

$$(f)^0_U E_U (g)^0_U \iff \{i \in \kappa | f(i) \in g(i)\} \in U. \tag{4.15}$$

Ultrapower Models

Then the ultrapower can be defined as

$$\mathrm{Ult}(V, U) := \langle {}^{\kappa}V \setminus U, E_U \rangle. \tag{4.16}$$

The version of Łoś' Theorem 25 that we will use is this: For any formula ϕ, and $\vec{f}$ from $^{\kappa}V$,

$$\mathrm{Ult}(V, U) \models \phi((f_0)^0_U, \ldots, (f_n)^0_U) \iff \{i \in \kappa | \phi[f_1(i), \ldots, f_n(i)]\} \in U. \tag{4.17}$$

Well-Foundedness

Recall that an ultrafilter U on a set S is called U *κ-complete* when U is closed under intersection of less than κ-many elements, an in particular σ-complete when it is ω_1-complete. A further condition on U insures that $\mathrm{Ult}(V, U)$ will lead to a well-founded model:

Proposition 26. *If U is σ-complete ultrafilter, then* $\mathrm{Ult}(V, U)$ *is well-founded.*

Proof. Assume $\mathrm{Ult}(V, U)$ is not well-founded, so that there is some infinitely descending sequence $(f_0) \ni (f_1) \ni (f_2) \ni \ldots$. This means that the set $\{\gamma | f_j(\gamma) \ni f_{j+1)}(\gamma)\}$ is in U for any j. Hence by σ-completeness the intersection is in U and so is non-empty. That means that $f_0(\gamma) \ni f_1(\gamma) \ni \ldots$ for some γ, contradicting the well-foundedness of V. □

Assuming U is ω_1-complete, we have by the preceding Lemma and Mostowski's Collapsing Lemma that there is a transitive class M_U with an isomorphism $\pi_U : \mathrm{Ult}(V, U) \to \langle M_U, \beta \rangle$. By Łoś' Theorem (4.17), and since the axioms of set theory hold in V, we have that M_U is an inner model.

The Canonical Ultrapower Elementary Embedding j

We define $[f]_U := \pi_U((f)^0_U)$, for π the isomorphism given by the Collapsing Lemma. We will also denote this as $[f]$ when it is clear which ultrafilter we have in mind. For any x, let f_x be the constant function $S \to \{x\}$.

The map defined by

$$j_U(x) := [f_x]_U \qquad \text{for } x \in V \tag{4.18}$$

is an elementary embedding of V into $\mathrm{Ult}(V, U)$ by Łoś' Theorem (4.17), and will be called the *canonical elementary embedding of V into the ultraproduct* $\mathrm{Ult}(V, U)$. This will be summed up as $j : V \prec M_U \cong \mathrm{Ult}(V, U)$. We will call models $\mathrm{Ult}(V, U)$ obtained in this way *ultraproduct models of ZFC generated by U*, or simply *ultraproduct models*.

4.3 Large Cardinals, Elementary Embeddings

In this section we show the connection between elementary embeddings and large cardinals. The key idea in this subsection is that large cardinals give ultrafilters which drive the ultraproduct model of V construction described above.

Assume that we have a measurable cardinal κ. The non-trivial, κ-additive measure on κ gives a non-principal, κ-complete ultrafilter U on κ. Let j be the canonical elementary embedding from V into the ultraproduct $\mathrm{Ult}(V, U)$ of V with respect to U, along with the map π from $\mathrm{Ult}(V, U)$ to its Mostowski collapse M. We show that j is non-trivial.

4.3.1 Large Cardinals to Elementary Embeddings

Proposition 27. *If there is a measurable cardinal κ, there is a non-trivial elementary embedding of V into an inner model M of* ZFC*, and the critical point of j is κ.*

Proof. The inner model M is the ultraproduct of V by U, the ultrafilter that witnesses that κ is measurable. The above development shows that the canonical ultraproduct embedding is elementary between V and the inner model $\mathrm{Ult}(V, U)$. We prove that κ is the critical point by showing that both $j(\alpha) = \alpha$ for any $\alpha < \kappa$, and then that $j(\kappa) > \kappa$ by squeezing an explicit element of M in between the two.

First assume for contradiction that α is least such that $j(\alpha) > \alpha$. Let $[f] = \alpha$ in $\mathrm{Ult}(V, U)$. Then $[f] < [c_a]$ in $\mathrm{Ult}(V, U)$, which means that $\{\gamma < \kappa | f(\gamma) < \alpha\}$ is in U. By the κ-completeness of U, then $\{\gamma < \kappa | f(\gamma) = \beta\} \in U$ for some $\beta < \alpha$. So we have that $[f] = j(\beta)$, and hence $\beta < \alpha = [f] = j(\beta)$. Contradiction with α the minimal ordinal sent upwards by j.

Now consider the identity map id on κ and $[id]$ in $\mathrm{Ult}(V, U)$. Every bounded subset of κ has measure 0, so that for any $\gamma < \kappa$, $c_\gamma(\gamma) < id(\gamma)$ on a set of measure 1. Hence $\gamma = j(\gamma) < [id]$ for every γ less than κ, and it follows that $\kappa \leq [id]$. Since also $id(\gamma) < \kappa$ for every γ, then $[id] < j(\kappa)$, so combining, we have $\kappa \leq [id] < j(\kappa)$.

Hence if there's a measurable cardinal κ, then there's an elementary embedding between V and the ultraproduct of V by U, and κ is the least ordinal moved by j, or its critical point. □

4.3.2 Elementary Embeddings to Large Cardinals

Conversely, for any non-trivial elementary embedding between inner models, the least ordinal moved must be a measurable cardinal.

Theorem 28 (Keisler, 1962)**.** *For any $j : M \prec N$ with $j \neq id$, we have that the least ordinal κ such that $\kappa < j(\kappa)$ has a non-principal κ-complete ultrafilter on it. Hence if there is a non-trivial elementary embedding j of V to some inner model M of* ZFC*, then there is a measurable cardinal.*

Proof. The strategy of the proof is, for any such j, M, N, to construct a non-principal κ-complete ultrafilter U_j on κ so that κ is measurable. We put

$$X \in U_j \iff X \subseteq \kappa \wedge \kappa \in j(X). \tag{4.19}$$

We check the properties of a κ-complete non-principal ultrafilter on U_j:

1. $\emptyset \notin U$ if and only if $\kappa \notin j(\emptyset)$, but as $\emptyset$ is a first-order definable set, $j(\emptyset) = \emptyset$, so that we have $\kappa \notin \emptyset$, and hence $\kappa \notin j(\emptyset)$.

2. $\kappa \in U$ if and only if $\kappa \in j(\kappa)$ if and only if $\kappa < j(\kappa)$, which holds by assumption that κ is a non-fixed point of j.

3. Towards superset closure, assume $X \in U, X \subseteq Y \subseteq \kappa$. $X \in U$ is equivalent to $\kappa \in jX$. Then as $X \subseteq Y$ and $\subseteq$ is first-order definable, then $j(X) \subseteq j(Y)$. Hence $\kappa \in jY$, so that $Y \in U$, as desired.

4. For "ultra," let $X \in \kappa$. We would like $\kappa \in j(X) \vee \kappa \in j(\kappa) - j(X)$. But $\kappa \in j(\kappa)$. Then $\kappa \in S \vee \kappa \in j(\kappa) - S$ holds for any set S, in particular $j(X)$.

5. For non-principality, we claim that for all $\beta \in \kappa$, we have $\{\beta\} \notin U$. We have that $\{\beta\} \notin U$ is equivalent to $\kappa \notin j(\{\beta\})$. But we have that $\{\beta\}$ is a definable set in terms of β and that $\beta = j\beta$ (by assumption $\beta < \kappa$). Hence $\{\beta\} = j\{\beta\}$. Therefore we have that $\kappa \notin j(\{\beta\})$ (for otherwise the contradiction that $\kappa \in \{\beta\}$ for some $\beta \in \kappa, \kappa > \omega$). Equivalently $\{\beta\} \notin U$.

6. We claim κ-completeness: Assume for each $\iota < \beta < \kappa$, we have $X_\iota \subseteq \kappa$ and $X_\iota \in U$. Denote by F the function with domain β listing these X_ι's, and use the notation $F(\iota)$ to denote X_ι. Then $\kappa \in \bigcap_{\iota<\beta}\{j(F(\iota))\}$. We claim that j commutes with any function with domain an ordinal less than κ, which for simplicity we state for F:

   Proposition 29. *For any $\iota < \kappa$, we have:*

   $$j(F(\iota)) = j(F)(\iota). \tag{4.20}$$

   In particular,

   $$\bigcap_{\iota<\beta}\{j(F(\iota))\} = \bigcap_{\iota<\beta}\{j(F)(\iota))\}. \tag{4.21}$$

   Proof. The difference between the two sides of the first equation of the proposition is that the left is the result of applying j to the set $F(\iota)$, whereas the right is the result of first applying j to the whole family F and *then* taking the ι'th value. (Note that as F is a function defined for values $\iota < \beta$, then by elementarity, jF will also be a function defined for values $\iota < j\beta = \beta$.) The proposition follows by considering the first-order formula $\phi(x, y, z)$ formalizing "x is the value on input y of the function z":

   We have $V \models \phi(F(\iota), \iota, F)$, so that $N \models \phi(j(F(\iota)), j\iota, jF)$ by elementarity, whence $N \models \phi(j(F(\iota)), \iota, jF)$ as $\iota < \kappa$. But this just means that $j(F(\iota)$ is the value on input ι of the function $j(F)$. Equivalently, $j(F(\iota)) = j(F)(\iota)$, as desired. □

   We also have the following:

Proposition 30. $\bigcap_{\iota<\beta}\{j(F)(\iota)\} = j(\bigcap_{\iota<\beta}\{F(\iota)\})$.

Proof. This follows from the distributivity of j over less than κ many intersections or unions, the proof of which uses reasoning similar to that of the proposition above. □

Combining the two above equalities, we have $\bigcap_{\iota<\beta}\{j(F(\iota))\} = j(\bigcap_{\iota<\beta}\{F(\iota)\})$. From our assumption that κ is in each $j(F(\iota))$, we have that $\kappa \in \bigcap_{\iota<\beta}\{j(F(\iota))\}$. Hence $\kappa \in j(\bigcap_{\iota<\beta} F(\iota))$ by substitution. Therefore $\bigcap_{\iota<\beta} F(\iota) \in U$. As $F(\iota)$ was a function that lists each X_ι, we have that $\bigcap_{\iota<\beta} X_\iota \in U$. This completes the proof of κ-completeness of U.

Hence U is an κ-complete non-principal ulrafilter on κ, so that κ is by definition measurable. This completes the proof of the Theorem. □

4.4 Structural Results on Ultrapower Models

Not every elementary embedding will embed into an ultrapower model, by for those that do, we have rich additional information about the embedding and the inner models.

Ultraproduct models have the very nice representation property. The model $Ult(V, U)$ can be represented as the closure of $\{\kappa\}$ by functions of the form $j_U(f)$.

Proposition 31. *Suppose U is a normal ultrafilter over κ. Then*

$$M_U = \{\, j_U(f)(\kappa) \mid f : \kappa \to \mathrm{V} \,\}$$

Proof. For $\subseteq$ assume $x \in M_U$, so that $x = [f]_U$ for some $f : \kappa \to V$. We claim $j_U(f)(\kappa) = j_U(f)([d]_U) = [f]_U$: the first equation is substitution of $\kappa = [d]$, whereas the last holds as $j_U(f) = [c_f]_U$ for c_f, the constant f function. Then $[f]_U = [c_f]_U([d]_U)$ is equivalent to $(c_f)_U((d)_U) = (f)_U$, whence by Łoś for $\phi(v_1, v_2, v_3)$ defining "v_1 applied to v_2 is v_3," is equivalent to $f(\alpha) = f(\alpha)$ ultrafilter everywhere. $\supseteq$ is clear. □

We also have the following collection of properties of ultraproduct models that we collect into one proposition:

Proposition 32 (Structure Theory of Ultrapower Models)**.** *Assume that the ultrafilter U witnesses that an uncountable κ is measurable.2 For $j : V \to \mathrm{Ult}(V, U) \cong M$, we have:*

a. $x = jx$ *for any* x *of rank less than* κ*,*

b. $V_\kappa = (V_\kappa)^M$*,*

c. for any $X \in V_{\kappa+1}$*,* $jX \cap V_\kappa = X$*,*

d. $V_{\kappa+1} = (V_{\kappa+1})^M$*,*

e. $(M)^\kappa \subseteq M$*,*

f. $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$

g. $U \notin M$*.*

4.5 A Defect and a Goal, Towards Stronger Closure Properties

In the preceding sections, we introduced measurable cardinals , shown in Theorem 19 that they are strongly inaccessible, and proved as Theorems 27 and 28 what we call the *Fundamental Theorem of Measurable Cardinals.* In Section xx, we demonstrated a number of structural properties of ultrapower models. But it will be the limitations of these ultrapower models, that $U \notin M$ and U is only closed under κ-length sequence, that will become of interest to us moving forward. These deficiencies naturally suggest stronger axioms of infinity asserting the existence of elementary embeddings into models that do satisfy these stronger closure properties. Central to our concerns will be the following notion: an uncountable cardinal κ is *2-strong* when it is the critical point of a $j : V \prec M$ such that $V_{\kappa+2} \subseteq M$. The goal of construction an L-like model that is highly ordered, yet contains a 2-strong cardinal, will prove to be highly non-trivial. Analyzing the origin of this difficulty will preoccupy us for the remainder of the thesis.

Chapter 5

$L[U]$ and Iterated Ultraproducts

This chapter will analyze the model $L[U]$ of sets constructible relative to a measure U, and describe Kenneth Kunen's theorems on the uniqueness of $L[U]$.

5.1 The Model $L[U]$

For κ a measurable cardinal, and U an ultrafilter that witnesses measurability, we have that $L[U] = L[\bar{U}]$, where $\bar{U} = U \bigcap L[U]$. We have that $L[U]$ sees that U is a κ-complete non-principal ultrafilter, and that U is normal, if U is normal:

Proposition 33. *$L[U] \models U$ is a κ-complete non-principal ultrafilter, and if U is normal, then $L[U] \vdash U$ is normal.*

Proof. For normality, assume that f is a regressive function on κ. Since U is normal, then there is a $\gamma < \kappa$ such that the set $S = \{\alpha < \kappa | f(\alpha) = \gamma\}$ is in U. Since $S \in L[U]$, then $L[U] \models$ f is constant on some $\bar{U}$-large set. The other parts follow by similar arguments. $\square$

We now show that for sets of ordinals A, the GCH holds inside of $L[A]$ for sufficiently large α, irregardless of what kind of set A is.

Proposition 34. *Under $V = L[A]$, if α is such that $A \subset P(\omega_\alpha)$, then $L[A] \models \aleph_{\alpha+1} = 2^{\aleph_\alpha}$.*

Proof. Assume that X is a subset of ω_α. We can always find a cardinal λ so that both $A, X \in L_\lambda[U]$. Let M be an elementary submodel of $L_\lambda[U]$ so that $A, X \in M$, $\omega_\alpha \subset M$, and $|M| = \aleph_\alpha$, for instance by taking a Skolem Hull. Taking π to be the Mostowski transitive collapse map and letting $N = \pi(M)$, we have that $\pi(Y) = Y$ for every $Y \subset \omega_\alpha$ that is in M, since $\omega_\alpha \subset M$. As a specific case, we have that $\pi(X) = X$, and furthermore that $\pi(A) = \pi(A \bigcap M) = \pi(A) \bigcap N$. By the Condenstation Lemma for $L[A]$, we have that N is actually $L_\beta[A \bigcap N]$ for some β, so that $N = L_\beta[A]$. We have that $\beta < \omega_{alpha+1}$, since the size $|N|$ of N was built to be $\aleph_\alpha$. Hence $\omega_\alpha \supset X \in L_{\omega_{\alpha+1}}[A]$, so that we have $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ as desired. □

We now prove that the full GCH holds in $L[N]$ for N a normal measure on a measurable cardinal:

Theorem 35. *Under $V = L[D]$, the GCH holds, for D a normal measure on κ.*

5.2 Iterated Ultrapowers

We now wish to iterate the ultrapower operation. Assume that we have a measurable cardinal κ, and U, a κ-complete non-principal ultrafilter to witness this, and let $M_0 = V = Ult^{(0)}$. We can construct the ultraprower $Ult_U(V)$ as usual. This model with be well-founded, so we can identify M_1 with the transitive collapse of this ultraproduct, so that

$$M_1 = Ult_U(V) = Ult^{(1)}.$$

We let $j^{(0)}$ be the usual canonical embedding from $V = M_0$ into M_1, and define

$$\kappa^{(1)} = j^{(0)}(\kappa), U^{(1)} = j^{(0)}(U).$$

Since j is elementary, $\kappa^{(1)}$ will be a measurable, and $U^{(1)}$ a $\kappa^{(1)}$-complete ultrafilter to witness this. So we can now take the ultraproduct of M_1 modulo U_1, and define $\kappa^{(2)} = j^{(1)}(\kappa), U^{(2)} = j^{(1)}(U)$ as above. We can iterate this as many finte amount of times as we like, so that we have

$$Ult^{(1)}, \quad Ult^{(2)}, \quad \ldots, \quad Ult^{(n)}, \quad \ldots \qquad \text{for } n < \omega.$$

We note that each of the class-sized elements of this sequence is definable in V, since the initial segment of $Ult^{(n)}$ intersect with V_α is built from a sufficiently large V_β. For any $m < n$, we define the map $i_{m,n} : U^{(m)} \to U^{(n)}$ as the composition of the individual j maps:

$$i_{m,n} = j^{(n-1)} \circ j^{(n)} \circ \cdots \circ j^{(m+1)} \circ j^{(m)}(x) \qquad \forall x \in Ult^{(m)}.$$

This makes a commutative system, i.e. for any $m < n < p$, we have that

$$i_{m,p} = i_{n,p} \circ i_{m,n}.$$

We define $\kappa^{(n)}$ as the i^{th} image of κ, so that $\kappa^{(n)} = i_{0,n}(\kappa)$ and similarly $U^{(n)} = i_{0,n}(U)$. We have that for any n, both

$$\kappa^{(0)} < \kappa^{(1)} < \cdots < \kappa^{(n)} < \ldots$$

and that

$$Ult^{(0)} \supset Ult^{(1)} \supset \cdots \supset Ult^{(n)} \supset \ldots.$$

This makes a directed system $\{Ult^{(m)}, i_{m,n} | m, n \in \omega\}$ of ZFC-models and elementary maps. By our previous disucssion on directed systems, the direct limit $Ult^{(\omega)}$ is well-defined, and the limit maps $i_{m,\omega}$ are elementary, so that in particular, $Ult^{(\omega)}$ is a ZFC model.

We define $\kappa^{(\omega)} = i_{0,\omega}(\kappa)$ and $U^{(\omega)} = i_{0,\omega}(U)$. Even though in general V will not satisfy that $U^{(\omega)}$ is a $\kappa^{(\omega)}$-complete ultrafilter, the model $Ult^{(\omega)}$ will, so that we can construct the ultraproduct of $Ult^{(\omega)}$ modulo $U^{(\omega)}$, which we call $Ult^{(\omega+1)}$. We let $j^{(\omega)}$ be the canonical embedding from $Ult^{(\omega)}$ into this ultraproduct.

Letting $i_{\omega,\omega+1}$ be this canonical embedding, all the maps so far commute, that is, for $n < \omega$, we have

$$i_{n,\omega+1} = i_{\omega,\omega+1} \circ i_{n,\omega}.$$

Continuing in this way, we define the *iterated ultraproduct of* V as

$$(Ult^{(0)}, E^{(0)}) = (V, \epsilon)$$

$$(Ult^{(\alpha+1)}, E^{(\alpha+1)}) = \text{the ultraproduct of } (Ult^{(\alpha)}, E^{(\alpha)}) \text{ mod } U^{(\alpha)}$$

$$(Ult^{(\gamma)}, E^{(\gamma)}) = \text{the direct limit of } (Ult^{(\alpha)}, E^{(\alpha)}) \text{ and maps } i_{\alpha,\beta}, \text{ for } \alpha < \beta < \gamma$$

In general, for M a transitive model of ZFC, U a κ-complete ultrafilter in the sense of M, let $Ult_U^{(\alpha)}(M)$ be the α^{th} iterated ultraproduct of M.

It is unknown for us at this point whether the iterated ultraproduct is well-founded, but if it is, we will identify the well-founded ultraproduct with its transitive collapse.

Having defined the iterated ultraproduct, we can show the Factoring Lemma for Iterated Ultraproducts:

Lemma 36 (Factoring Lemma for Iterated Ultraproducts)**.** *If U^α is well-founded, then for each β, $Ult^{(\beta)}_{U^{(\alpha)}}(Ult^{(\alpha)})$ is isomorphic to $Ult^{(\alpha+\beta)}$, via some map e^α_β.*

Moreover, there are isomorphisms $e^{(\alpha)}_\xi$, $e^{(\alpha)}_\eta$ so that the following diagram commutes:

$$Ult^{(\xi)}_{U^{(\alpha)}}(Ult^{(\alpha)}) \xrightarrow{i^{(\alpha)}_{\xi,\eta}} Ult^{(\eta)}_{U^{(\alpha)}}(Ult^{(\alpha)})$$

$$\downarrow e^{(\alpha)}_\xi \qquad \downarrow e^{(\alpha)}_\eta$$

$$Ult^{(\alpha+\xi)}_U \xrightarrow{i_{\alpha+\xi,\alpha+\eta}} Ult^{(\alpha+\eta)}_U$$

where $i^{(\alpha)}_{\xi,\eta}$ is the embedding of $Ult^{(\xi)}_{U^{(\alpha)}}$ into $Ult^{(\eta)}_{U^{(\alpha)}}$

Proof. We show the existence of the $e^{(\alpha)_\beta}$ maps by induction on β. For $\beta = 0$, then $Ult^{(0)}_{U^{(\alpha)}}(Ult^{(\alpha)})$ is just $(Ult^{(\alpha)})$, so we let $e^{(\alpha)}_0$ be the identity. For β a successor ordinal, assume that we have the diagram up to β, in particular the map $e^{(\alpha)}_\beta$ between $Ult^{(\beta)}_{U^{(\alpha)}}(Ult^{(\alpha)})$ and $Ult^{(\alpha+\beta)}$. The model $Ult^{(\beta+1)}_{U^{(\alpha)}}(Ult^{(\alpha)})$ in the upper-right hand side of the diagram is the ultraproduct of the upper-left $Ult^{(\beta)}_{U^{(\alpha)}}(Ult^{(\alpha)})$ modulo $i^{(\alpha)}_{0,\beta}(U^{(\alpha)})$, while the model in the lower-right is the ultraproduct of the lower-left model modulo $U^{(\alpha+\beta)}$. By induction $e^{(\alpha)}_\beta(i^{(\alpha)}_{0,\beta}(U^{(\alpha)})) = U^{(\alpha+\beta)}$, so that the map $e^{(\alpha)}_\beta$ induces the isomorphism $e^{(\alpha)}_{\beta+1}$ between the next two models in the diagram. The limit case follows by a similar argument.

□

Chapter 6

Strong Cardinals, Extenders

6.1 Extenders

Suppose that N, M are inner models of ZFC and $j : N \prec M$. If j is nontrivial, there is a least ordinal $\mathrm{crit}(j)$ such that $j(\kappa) > \kappa$.

$$\begin{array}{ll} \text{We let} & \lambda \text{ be above } \kappa = \mathrm{crit}(j), \text{ and} \\ & \zeta \text{ the least such that } \lambda \leq j(\zeta) \ . \end{array}$$

For each $a \in [\lambda]^\omega$, define E_a as:

$$X \in E_a \quad \text{iff} \quad X \subseteq [\zeta]^{|a|} \cap N \wedge a \in j(X),$$

so that, although E_a need not be in N, we have $\langle N, \varepsilon, E_a \rangle \models E_a$ is a κ-complete ultrafilter over $[\zeta]^{|a|}$.

Now we specify

$$E := \langle E_a | a \in [\lambda]^\omega \rangle \text{ is the } (\kappa, \lambda)\textit{-extender derived from } j.$$

For $a \in [\lambda]^{<\omega}$, let

$$j_a : N \prec \mathrm{Ult}(N, E_a)$$

be the ultrapower embedding generated by E_a.

Let $(f)^0_{E_a} \in \mathrm{Ult}(N, E_a)$ be the least rank equivalence class of a function $f \in {}^{[\zeta]^{|a|}}N \cap N$, and let

$$k_a((f)^0_{E_a}) = j(f)(a) \ .$$

It is not too hard to check that k_a is elementary and

$$k_a \circ j_a = j.$$

As each E_a is ω_1-complete, we have that each $\mathrm{Ult}(N, E_a)$ and so has a transitive collapse M_a, so that $\mathrm{Ult}(N, E_a)$ and M_a will be identified.

Now we define maps i_{ab} between the M_a's. For $a \subseteq b$ both in $[\lambda]^{<\omega}$, when $b = \{\alpha_1, \ldots, \alpha_n\}$ with the convention that $\alpha_1 < \cdots < \alpha_n$, and $a = \{a_{i_1}, \ldots, \alpha_{i_m}\}$ such that $1 \leq i_i < \cdots < i_m \leq n$, we define projections $\pi_{ba} : [\zeta]^n \to [\zeta]^m$ by:

$$\pi_{ba}(\{\xi_1, \ldots, \xi_n\}) = \{\xi_{i_1}, \ldots, \xi_{i_n}\}.$$

It is routine to verify that $i_{ab} : M_a \to M_b$ defined by

$$i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}$$

is elementary, and that the maps commute:

$$i_{ab} \circ j_a = j_b, \qquad k_b \circ i_{ab} = k_a.$$

This makes $\langle \langle M_a | a \in [\lambda]^{<\omega} \rangle, \langle i_{ab} | a \subseteq b \rangle \rangle$ a directed system, so we specify

$$\langle M_E, \varepsilon_E \rangle \text{ is the direct limit,}$$

and

$$\begin{aligned} j_E : \langle N, \in \rangle &\prec \langle M_E, \in_E \rangle \ , \\ k_{aE} : \langle M_a, \in \rangle &\prec \langle M_E, \in_E \rangle \ , \text{ and} \\ k_E : \langle M_E, \in_E \rangle &\prec \langle M, \in \rangle \end{aligned}$$

the corresponding embeddings so that for any $a \in [\lambda]^{<\omega}$:

$$k_E \circ j_E = j \ , \ k_{aE} \circ j_a = j_E \ , \text{ and } k_E \circ k_{aE} = k_a \ .$$

We have that $\langle M_E, \in_E \rangle$ is well-founded, so we can assume

$$M_E \text{ is transitive and } \in_E = \in \cap (M \times M).$$

At this point we can prove:

Lemma 37. *1.* $M_E = \{j_E(f)(a) | a \in [\lambda]^{<\omega} \wedge f \in {}^{[\zeta]^{|a|}}N \cap N\}$.

2. For any γ *such that* $|V_\gamma|^M \leq \lambda$*, we have* $V_\gamma^M \subseteq \mathrm{ran}(k_E)$*,* $V_\gamma^{M_E} = V_\gamma^M$*, and* $k_E(x) = x$ *for any* $x \in V_\gamma^{M_E}$*.*

3. *We have* $\mathrm{crit}(k_E) \geq \lambda$*, so that* $\mathrm{crit}(j_E) = \kappa$ *and* $\lambda \leq j_E(\zeta)$*. When* $\lambda = j(\zeta)$*, then* $\mathrm{crit}(k_E) > \lambda$*, and so* $\lambda = j_E(\zeta)$*.*

Proof. See [1, 354]. □

The following, somewhat technical, definition defines extenders directly:

Definition 38. For N a inner model of ZFC, κ an N-cardinal, $\lambda > \kappa$, and $E = \langle E_a | a \in [\lambda]^{<\omega} \rangle$, E is an N-(κ, λ)-*extender iff* for some $\zeta \geq \kappa$:

1. For each $a \in [\lambda]^{<\omega}$, $\langle N, \in, E_a \rangle \models E_a$ is a κ-complete ultrafilter over $[\zeta]^{|a|}$
2. For at least one $a \in [\lambda]^{<\omega}$, $\langle N, \in, E_a \rangle \models E_a$ is not κ^+-complete.
3. For each $\xi < \zeta$, there is an a such that $\{s \in [\zeta]^{|a|} | \xi \in s\} \in E_a$.
4. (Coherence) For any $a \subseteq b$ both in $[\lambda]^{<\omega}$ and $\pi_{ba} : [\zeta]^{|b|} \to [\zeta]^{|a|}$ is a projection so that for $b = \{\alpha_1, \ldots, \alpha_n\}$ and $a = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$, $\pi_{ba}(\{\zeta_1, \ldots, \zeta_m\}) = \{\zeta_{i_1}, \ldots, \zeta_{i_m}\}$. Then
$$X \in E_a \iff \{s | \pi_{ba}(s) \in X\} \in E_b \ .$$
5. (Well-foundedness) Whenever $a_m \in [\lambda]^{<\omega}$ and $X_m \in E_{a_m}$ for $m \in \omega$, then there is a $d : \bigcup_m a_m \to \zeta$ such that $d``a_m \in X_m$ for every m.
6. (Normality) Whenever $a \in [\lambda]^{<\omega}$, $f \in {}^{[\zeta]^{|a|}}N \cap N$, and
$$\{\, s \in [\zeta]^{|a|} \mid f(s) \in \max(s) \,\} \in E_a \ ,$$
there is a $b \in [\lambda]^{<\omega}$ with $a \subseteq b$ such that (with π_{ba} as before)
$$\{\, s \in [\zeta]^{|b|} \mid f(\pi_{ba}(s)) \in s \,\} \in E_b \ .$$

Extender Premice

Definition 39. We call $\langle N, \in, E \rangle$ a ZFC^- *extender-premouse (at* κ, λ*)* or just an *extender premouse* when E is an N-(κ, λ) extender and $N = L_\zeta[E]$ for some ζ (allowing $N = L[E]$).

6.2 Strong Cardinals

We define:

κ is *γ-strong* *iff* there is a $j : V \prec M$ such that:
(a) $\mathrm{crit}(j) = \kappa$ and $\gamma < j(\kappa)$, and
(b) $V_{\kappa+\gamma} \subseteq M$.

κ is *strong* *iff* κ is γ-strong for every γ .

Theorem 40. *[(a)]*

1. *κ is γ-strong iff there is a (κ, λ)-extender E such that $V_{\kappa+\gamma} \subseteq M_E$, $\gamma < j_E(\kappa)$, and $\lambda > |V_{\kappa+\gamma}|$.*

2. *κ is $\gamma + 1$-strong iff for some $\lambda > \kappa$, there is a (κ, λ)-extender E such that $V_{\kappa+(\gamma+1)} \subseteq M_E$. Hence κ is strong iff for any set x, there is a $j : V \prec M$ with $\mathrm{crit}(j) = \kappa$ and $x \in M$.*

Proof. (a) ($\Leftarrow$) By Lemma 37 (c) above, $\mathrm{crit}(j_E) = \kappa$, so it is clear that j_E, M_E give us that κ is γ-strong. ($\Rightarrow$) Assume $j : V \prec M$ with $V_{\kappa+\gamma} \subseteq M$, i.e. $V_{\kappa+\gamma} \subseteq V^M_{\gamma+\kappa}$. We verify that the extender E_j derived from j of length $|V_{\kappa+\gamma}|^+$ works. By Lemma 37 (b), $V^{M_E}_{\kappa+\gamma} = V^M_{\kappa+\gamma}$, so that $V_{\kappa+\gamma} \subseteq V^{M_E}_{\kappa+\gamma}$, and hence $V_{\kappa+\gamma} \subseteq M_E$. Now assume for contradiction $\gamma \geq j_E(\kappa)$, so that by elementarity, $k(\gamma) \geq k \circ j_E(\kappa) = j(\kappa)$, where k is the natural map from Ult_E. But $\mathrm{crit}(k) \geq \lambda = |V_{\kappa+\gamma}|^+ > \gamma$, where the first equality is by the same Lemma. Hence $k(\gamma) = \gamma$, so that $\gamma \geq j(\kappa)$, which contradicts that κ is γ-strong.

(b) ($\Rightarrow$) is immediate from (a). ($\Leftarrow$) Assume E with $V_{\kappa+(\gamma+1)} \subseteq M_E$, and consider $j_E : V \prec M_E$; we need to get $\gamma < j(\kappa)$ for some j. As in Proposition 23.15 of [1], we use the fact that $\gamma < \sup\{\, j^n(\kappa) \mid n \in \omega \,\}$, where $j^n(x)$ is j composed n times. As $\langle j^n(\kappa) | n \in \omega \rangle$ is strictly increasing, then $\gamma < j^k(\kappa)$ for some $k \in \omega$. It will suffice that for every $n \in \omega$, $n > 0$, there is a ZFC-model M_n such that $j^n : V \prec M_n$ and $V_{\kappa+\gamma} \subseteq M_n$. By induction, for $n = 1$, M_E works. Extending j to classes C, define $j^+(R) = \bigcup_\alpha j(R \cap V_\alpha)$. Omitting details, $M_{n+1} = j^+(M_n)$ will work for inductive steps.

For “hence”, from the preceding, κ is strong if and only if for any $\gamma \in \mathrm{On}$, exists $j : V \prec M$ such that $\mathrm{crit}(j) = \kappa$ and $V_{\kappa+\gamma} \subseteq M$. Rightwards then, for any set x, κ is $(\mathrm{rank}(x) + 1)$-strong, giving the required $j : V \prec M$. Conversely, letting $\gamma \in \mathrm{On}$, $V_{\kappa+\gamma}$ is a set, giving $j : V \prec M$ witnessing that κ is γ-strong. □

Let us denote by $M \prec_n N$ that Σ_n formulas are absolute between M and N. It is a known result by Levy that for κ uncountable, $H_\kappa \prec_1 V$, where H_κ is the set of sets hereditarily of cardinality less than κ.

Proposition 41. *If κ is strong, then $V_\kappa \prec_2 V$.*

Proof. ($\Rightarrow$) Let ϕ be Σ_2, for simplicity $\exists v_1 \psi[v_1]$ for a Π_1 ψ. Assume $V_\kappa \models \phi(x)$ for $x \in V_\kappa$, so that $V_\kappa \models \psi[y, x]$ for some $y \in V_\kappa$. As κ is strongly inaccessible, $H_\kappa = V_\kappa$, and the aforementioned result gives $\psi[x, y]$.

($\Leftarrow$) Assume $\psi(y, x)$ for $x \in V_\kappa$. Letting $\alpha > \operatorname{rank}(y)$, as κ strong, there is a $j : V \prec M$ such that $\operatorname{crit}(j) = \kappa$, $V_\alpha \in M$ and $|V_\alpha| < j(\kappa)$. Then $y \in V_\alpha$, so $y \in V_{|V_\alpha|}$, $V_{|V_\alpha|} = (V_{|V_\alpha|})^M$ and hence $y \in (V_{j(\kappa)})^M$. Also $x = j(x) \in V_\kappa = (V_\kappa)^M$. Since $x, y \in (V_{j(\kappa)})^M$ and ψ is Π_1, then $V_{j(\kappa)} \models \psi[y, x]$. Hence $(V_{j(\kappa)} \models \psi[y, x])^M$, so that $V_\kappa \models \phi[x]$ by elementarity. $\square$

Chapter 7

Inner Models for Strong Cardinals

In this section we show some results related to inner models for strong cardinals.

7.1 No Strong Cardinals in L[U]

Proposition 42. *There are no strong cardinals in the model $L[E]$ with E an extender witnessing strongness.*

In fact a stronger property holds:

Lemma 43. *Suppose that there is a strong cardinal. Then $V \neq L[A]$ for any set A.*

Proof. For the Theorem, as a strong cardinal is γ-strong for every $\gamma \in \mathrm{On}$, it will clearly suffice to prove the following local version: Suppose that there is a γ-strong cardinal. Then $V \neq L[A]$ for any set A with $\mathrm{rank}(A) < \kappa + \gamma$.

Assume for contradiction that there is a γ-strong cardinal κ, and that $V = L[A]$ for a set $A \in V_{\kappa+\gamma}$. By our characterization of strongness, there is a $j_A : V \prec M$ such that $\mathrm{crit}(j) = \kappa$ and $A \in M$.

M contains all ordinal, M is a transitive model of ZFC, and $M \subseteq L[A] = V$. We claim $L[A] \subseteq M$ also. M satisfies the sentence "$V = L[X]$", so that it suffices to show $A \cap M \in M$, but this is clear as $A \in M$. Hence we have $j : V \prec V$, contradiction with Kunen's Theorem on no elementary embeddings from V to V. □

Note that this proof also works by assuming a proper class of measurable cardinals instead of a single strong cardinal. Since the measurables occur at any height, we are guaranteed elementary embeddings that fix arbitrary initial segments of the universe, just what a single strong cardinal provides.

Proof of our proposition. We can argue inside the model $L[E]$ as in Lemma above. □

Hence the approach of building a canonical inner model by constructing relative to a set witnessing a large cardinal property has to fail for full strongness. This result also gives insight on $L[U]$ models. For suppose κ is measurable, and let U witness this, so that $L[U]$ is a model of measurability. Then constructing relative to U is optimal with respect to the rank of the set chosen: We have that $U \in V_{\kappa+2}$, yet no set $A \in V_{\kappa+1}$ would model "there is a measurable".

With this result, we can prove a generalization of Scott's result that if there exists a κ measurable, then $V \neq L$:

Corollary 44. *If there exists a measurable κ, then $V \neq L[X]$ for any $X \in V_{\kappa+1}$, and in particular if there is a measurable, then $V \neq L[\emptyset] = L$.*

Proof. A cardinal is measurable if and only if it is 1-strong. □

7.2 Further Results

Lemma 45. *If $V = L[E]$ for some extender E, then there exist no measurable cardinals.*

Proof. (Sketch) Assume for contradiction that there is some measurable cardinal λ with ultrafilter U. Let $M = \mathrm{Ult}(V, U)$. It will be enough to show for a contradiction that $V = L[E] = M$, either by Kunen's theorem or noticing that U cannot be in $\mathrm{Ult}(V, U)$.

If $\lambda > \kappa$.

If $\lambda < \kappa$. **Show $j(E) = E \cap M$.** Then $M = L[j(E)] = L[E \cap M] = L[E]$. □

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